p-adic approach to differential equations

Problem set 1

Amice ring

For every prime number p, the Amice ring is defined as follows

$$\mathcal{A}_p = \left\{ \sum_{n\in\mathbb{Z}} a_n t^n : a_n \in \mathbb{Q}_p, \lim_{n\to-\infty} |a_n|_p = 0 \text{ and } \sup_{n\in\mathbb{Z}} |a_n|_p < \infty \right\}.$$

For every $f = \sum_{n\in\mathbb{Z}} a_n t^n$, we set

$$|f|_{\mathcal{G}} = \sup_{n\in\mathbb{Z}} |a_n|_p.$$

(i) Prove that $|\ |_{\mathcal{G}}$ is a norm. This norm is called the Gauss norm.

(ii) Prove that $\mathcal{A}_p$ is complete with respect to the Gauss norm.

(iii) Prove that $\mathbb{Q}(t) \subset \mathcal{A}_p$ and show that

$$\left| \frac{\sum_i a_i t^i}{\sum_j b_j t^j} \right|_{\mathcal{G}} = \frac{\max_i |a_i|_p}{\max_j |b_j|_p}.$$

Conclude that $E_p \subset \mathcal{A}_p$, where E_p is the p-adic closure of $\mathbb{Q}(t)$ called the field of p-adic analytic elements.

(iv) An element of $f = \sum_{n\in\mathbb{Z}} a_n t^n \in \mathcal{A}_p$ is invertible if and only if there is $n_0 \in \mathbb{Z}$ such that $|f|_{\mathcal{G}} = |a_{n_0}|$.

(v) If $f = \sum_{n\geq 0} a_n z^n \in \mathcal{A}_p$ has radius of convergence greater than 1 then $f \in E_p$.

Hypergeometric Frobenius structures

A generalized hypergeometric differential operator of order $n \geq 1$ is given by

$$L = (\theta + \beta_1 - 1)(\theta + \beta_2 - 1)\dots(\theta + \beta_n - 1) - t(\theta - \alpha_1)\dots(\theta - \alpha_n), \qquad \theta = t\frac{d}{dt}$$

with some complex numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$. This is a Fuchsian operator with singularities at $0, 1, \infty$. The local exponents read

$$\begin{aligned} &1-\beta_1, \dots, 1-\beta_n \qquad \text{at} \quad t=0,\\ &\alpha_1, \dots, \alpha_n \qquad \text{at} \quad t=\infty,\\ &1, 2, \dots, n-1, -1 + \sum_{i=1}^{n} (\beta_i - \alpha_i) \qquad \text{at} \quad t=1. \end{aligned}$$

The monodromy representation of L is known to be irreducible if and only if $\alpha_i - \beta_j \notin \mathbb{Z}$ for all i, j. In his thesis in 1961 Levelt gave a beautiful explicit proof of rigidity of monodromy groups of irreducible hypergeometric monodromy operators (see [1, §1.2]).

(i) Check that an irreducible hypergeometric differential equation satisfies Katz' criterion of rigidity:

Theorem(Nick Katz, see [1, §4]) Let $M_1, \ldots, M_r \in GL_n(\mathbb{C})$ be an irreducible tuple satisfying the relation $M_1 \cdot \ldots \cdot M_r = I$. Denote $\delta_i = \mathrm{codim}_{\mathbb{C}}\{A \in M_n(\mathbb{C}) : AM_i = M_i A\}$. Then

  (i) $\delta_1 + \ldots + \delta_r \geq 2(n^2 - 1)$,

  (ii) the tuple is rigid if and only if $\delta_1 + \ldots + \delta_r = 2(n^2 - 1)$.

To apply this to a Fuchsian differential equation one takes r to be the number of its singular points in $\mathbb{P}^1(\mathbb{C})$ and $M_1, \ldots, M_r$ to be monodromy transformations around simple loops around this points.

(ii) Suppose that $\alpha_i, \beta_j \in \mathbb{Q}$ and $\alpha_i - \beta_j \notin \mathbb{Z}$ for all i, j. Then the hypergeometric operator L satisfies the conditions of Daniel's Theorem on existence of p-adic Frobenius structure. Compute the order of this Frobenius structure and the set of primes for which it exists using the recipe given in the lecture.

p-adic analytic continuation

Let us consider the hypergeometric series

$$\mathfrak{f}(t) = {}_2F_1(1/2, 1/2, 1; t) = \sum_{n \geq 0} \frac{(1/2)_n^2}{n!^2} t^n.$$

Dwork has shown in his "p-adic cycles" paper that, for all $p > 2$, the quotient $\mathfrak{f}(t)/\mathfrak{f}(t^p)$ belongs to E_p. More precisely, he showed that for all $p > 2$ and $s \geq 1$

$$\frac{\mathfrak{f}(t)}{\mathfrak{f}(t^p)} = \frac{\mathfrak{f}_s(t)}{\mathfrak{f}_{s-1}(t^p)} \bmod p^s \quad \text{with} \quad \mathfrak{f}_s(t) = \sum_{n=0}^{p^s-1} \frac{(1/2)_n^2}{n!^2} t^n.$$

(i) Show that the p-adic radius of convergence of $\mathfrak{f}(t)/\mathfrak{f}(t^p)$ is 1 for any $p > 2$.

(ii) Consider the region

$$\mathcal{D} = \{y \in \mathbb{Z}_p : |\mathfrak{f}_1(y)|_p = 1\}$$

and check the following facts:

  (a) $\{y \in \mathbb{Z}_p : |y| < 1\} \subset \mathcal{D}$, and if $y \in \mathcal{D}$ then $y^p \in \mathcal{D}$;

  (b) for every $s \geq 0$ one has $|\mathfrak{f}_s(y)|_p = 1$ when $y \in \mathcal{D}$;

  (c) the sequence of rational functions $\mathfrak{f}_s(y)/\mathfrak{f}_{s-1}(y^p)$ converges uniformly in $\mathcal{D}$, and if we denote the limiting analytic function by $\omega(y) = \lim_{s\to\infty} \mathfrak{f}_s(y)/\mathfrak{f}_{s-1}(y^p)$ then for all $s \geq 1$

$$\sup_{y \in \mathcal{D}} \left| \omega(y) - \frac{\mathfrak{f}_s(y)}{\mathfrak{f}_{s-1}(y^p)} \right| \leq \frac{1}{p^s};$$

  (d) $\mathfrak{f}(t)/\mathfrak{f}(t^p)$ is the restriction of $\omega(t)$ to $\{y \in \mathbb{Z}_p : |y|_p < 1\}$.

Remark: The above procedure of analytic continuation allows to evaluate $\omega(y)$ at points $y \in \mathbb{Z}_p^\times$ such that $|\mathfrak{f}(y)|_p = 1$. Dwork also noted that the value $\omega(y_0)$ at a Teichmuller units $y_0 \in \mathbb{Z}_p^\times$, $y_0^{p-1} = 1$ is equal to the p-adic unit root of the elliptic curve $y^2 = x(x-1)(x-\overline{y}_0)$ where $\overline{y}_0$ is the reduction of y_0 modulo p. The condition $|f_1(y_0)|_p = 1$ chooses the ordinary elliptic curves in the Legendre family. A vaste generalisation of the above Dwork's congruences along with the evaluation of the respective p-adic analytic element is given in "Dwork crystals II" by Beukers-Vlasenko (see Theorem 3.2 and Remark 4.5).

(iii) Argue that the sequence of rational functions $\mathfrak{f}_s(t)/\mathfrak{f}_{s-1}(t^p)$ converges in the Gauss norm, and hence $\omega(t) \in E_p$

p-adic Frobenius structure for differential equations of rank 1

(i) Prove that, for any $p > 2$, the differential operator

$$\frac{d}{dt} - \frac{\mathfrak{f}'(t)}{\mathfrak{f}(t)}$$

has a p-adic Frobenius structure of period 1. Here f is the hypergeometric function considered in the previous set of exercises.

(ii) Let $L = d/dt - a(t)$ be a differential operator with $a(t) \in \mathbb{Q}(t)$. Prove that if L has a p-adic Frobenius structure for almost all primes p then $a(t) = f'(t)/f(t)$ with $f(t) \in \mathbb{Q}[[t]]$ algebraic over $\mathbb{Q}(t)$. Is the converse true?

Hint: Use the fact that the Grothendieck-Katz p-curvature conjecture holds for operators of rank 1.

(iii) Prove that the differential equation $d/dt - 1$ does not have a p-adic Frobenius structure for any p.

(iv) Let π_p be in $\overline{\mathbb{Q}}$ such that $\pi_p^{p-1} = -p$. Prove that $d/dt - \pi_p$ has a p-adic Frobenius structure.

Remark: A. Pulita in his work *Frobenius structure for rank one p-adic differential equations* gives a characterization of the differential operators of rank 1 having a p-adic Frobenius structure for given p.

References

[1] F. Beukers, *Hypergeometric functions of one variable*, notes from MRI spring school held in Groningen in 1999 `https://webspace.science.uu.nl/~beuke106/springschool99.pdf`

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Problem set 2

Sharif and Woodcock [1] proved that

$$\mathfrak{f}_r(t) = \sum_{n\geq 0} \binom{2n}{n}^r t^n$$

is transcendental over $\mathbb{Q}(t)$ for $r > 1$. In fact, they prove that for $r > 1$ the sequence $\{deg(\mathfrak{f}_{r|p})\}_{p\in\mathcal{P}}$ is not bounded and thus, $\mathfrak{f}_r(t)$ transcendental. The fact that $\mathfrak{f}_r$ is p-Lucas for all $p > 2$ is crucial in their argument for proving that such a sequence is not bounded. (See exercise 2 in Julien's sheet)

The goal of this problem session is to prove the transcendence of power series that are not p-Lucas for any p. For every $r \geq 1$, we consider the hypergeometric series

$$\mathfrak{g}_r(t) = \sum_{n\geq 0} \frac{-1}{2n-1} \binom{2n}{n}^r t^n \in 1 + t\mathbb{Z}[[t]].$$

(i) Prove that $\sum_{n\geq 0} a_n t^n \in 1 + t\mathbb{Z}[[t]]$ is p-Lucas if and only if, for all $m \geq 0$ and all $s \in \{0, \dots, p-1\}$,

$$a_{mp+r} = a_m a_r \bmod p.$$

(ii) Prove that, for all $r \geq 1$, $\mathfrak{g}_r(t)$ is not p-Lucas for any $p > 2$.

(iii) Prove that, for all $p > 2$, $\mathfrak{g}_r(t) = A_p(z)\mathfrak{f}_r(t)^p$, where $A_p(t) \in \mathbb{F}_p[t]$ has degree less than p.

(iv) Prove that, for all $p > 2$, $deg(\mathfrak{g}_{r|p}) = deg(\mathfrak{f}_{r|p})$.

(v) Conclude that $\mathfrak{g}_r(t)$ is transcendental over $\mathbb{Q}(t)$ for $r > 1$.

References

[1] H. Sharif and C. F. Woodcock *On the transcendence of certain series,* J. Algebra **121** (1989), 364–369.